Chaotic Langevin equation with the deterministic algebraically correlated noise

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We solve the Langevin equation, consisting of a double-well potential and a periodic time-dependent driving term, with a deterministic, algebraically correlated noise. Homoclinic instabilities are studied by means of the Melnikov method. The influence of noise on chaotic motion is discussed in terms of Lyapunov exponents. The results are compared with the case of exponentially falling noise correlations. A simple example of passage over the potential barrier is considered in the context of dynamical stability. [S1063-651X(99)05403-3]

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The Langevin description of a stochastic process, including the uncorrelated random force $f(t)$, is in many cases an unrealistic idealization. In fact, a process can be approximately regarded as Markovian if the time scales involved are large compared to the noise correlation time. Such an approximation must fail if one tries to describe a high-dimensional system using the Langevin formalism with only few degrees of freedom, because that procedure destroys in general the Markovian property of the original system [1]. Some physical problems impose specific requirements in the of matter correlations. For example, algebraic correlations (AC's), have been found in connection with the noise-induced Stark broadening [2] and molecular dynamics description of nuclear collisions [3]. In those problems, the stochastic force autocorrelation function $C_f(t)=\langle f(0)f(t)\rangle \sim 1/t$. This form of the noise autocorrelation function implies important consequences for the stochastic motion of the Brownian particle [4]. The diffusion coefficient is not constant but rises with time like $\ln t$; thus the transport is faster than in the case of rapidly decaying correlations (the anomalously enhanced diffusion). Trajectories of the Brownian particle are shaped typically for Lévy flights [5]. Moreover, the energy spectra of particles escaping from a potential well of parabolic shape have a Gaussian tail, and exhibit a pronounced sharp peak, whereas the survival probability depends on time like $1/t$. In most of the applications, however, we have to deal with potentials more complicated than the harmonic oscillator considered in Ref. [4], and systems with many degrees of freedom. Then the motion can become chaotic. The main purpose of this paper is to study the onset of chaos in the Langevin equation with AC noise.

A first token of irregularity in a dynamical system is the presence of homoclinic points. Stable and unstable manifolds connected to a given hyperbolic fixed point do not coincide, as is the case for an integrable system, but they themselves intersect transversely. As a result, the motion near separatrices becomes chaotic. A method introduced by Melnikov [6,7] allows us to identify the homoclinic instability in the first order of the perturbation theory. Bulsara, Schieve, and Jacobs [8] used it to study stability properties of nonlinear Langevin equations with white noise. We will apply the Melnikov method to a system consisting of a double-well potential, friction and a time-dependent periodic force (the Duffing oscillator [9]), driven additionally by the stochastic, algebraically correlated force $f(t)$. The Langevin equation reads

$$\dot{x}=v,$$
$$\dot{v}=x-x^3-\alpha v+\gamma\cos\omega t+f(t), \tag{1}$$

where we have taken the particle mass to be equal to 1. For different choices of constant parameters α, γ, and ω, the Duffing oscillator behaves either regularly or chaotically [10]. About the stochastic force $f(t)$ we assume that $\langle f(t)\rangle =0$ and $C_f\sim 1/t$, where $\langle\ \rangle$ denotes the average over a statistical ensemble. A physical process which meets these requirements, and therefore can be used as a noise generator in numerical simulations, is the Lorentz gas of periodically distributed scatterers, equivalent to the Sinai billiard with periodic boundary conditions. That deterministic, conservative, and strongly chaotic system consists of a particle with the velocity $\mathbf{u}=(u_1,u_2)$, bouncing elastically from hard disks and moving freely between successive collisions. The particle motion is determined by geometry of the billiard: the disk radius R_D and the distance between disk centers L_D. One can distinguish two important cases. If $L_D>2R_D$ (the open horizon), the particle can wander throughout the entire lattice, otherwise the disks intersect themselves and the particle is confined within a small area (the close horizon). In the case of open horizon the particle velocity autocorrelation function $C_u=\langle u_i(0)u_i(t)\rangle$, $i=1$ and 2 has the required algebraic form $1/t$, in the limit of long time [11]. This surprisingly long correlation tail is due to the slowly falling free path lengths distribution $\sim s^{-3}$ [12]. For the close horizon C_u is exponential (the EC case). Thus inserting the velocity of particle in the Lorentz gas system to the Langevin equation, one can simulate both kinds of correlation. In the following, we take $f(t)=ku_1(t)$ where k is the noise amplitude, providing $|\mathbf{u}|=1$. We have $ku_1(t)\equiv f_i$ $=$const for $t_i\leqslant t<t_{i+1}$ if collisions with disks take place at time t_i.

For general nonlinear systems, the stable and unstable separatrices usually do not join to each other smoothly but intersect themselves. The Melnikov idea consists of checking whether that intersection really happens, calculating directly the distance between the separatrices in the first order of the perturbation theory. To apply it to our problem, we rewrite the Langevin equation (1) in the form $\dot{v}=x-x^3+f(t)+\epsilon(\gamma\cos\omega t-\alpha v)$ with a small parameter ϵ. Therefore the periodic driving force and friction are treated as a perturbation. The noise is included in the unperturbed term and the unperturbed system is given by the Hamiltonian $H_0=V^2/2-X^2/2+X^4/4-f(t)X$. The Melnikov distance D is defined as a projection of a vector linking the stable and unstable separatrices at a given time, along a normal to the unperturbed orbit [7]. Let t_i $(\cdots<t_{-1}<t_0=0<t_1<\cdots)$ denote the times at which the stochastic force changes its value. Then D can be expressed as a function of an arbitrary initial time τ in the following way:

$$D(\tau)=-\gamma\cos(\omega\tau)I_1+\gamma\sin(\omega\tau)I_2+\alpha I_3. \tag{2}$$

The integrals $I_1=\Sigma_{i=-\infty}^{\infty}\int_{t_i}^{t_{i+1}}V_i(t)\cos\omega t\,dt$, $I_2=\Sigma_{i=-\infty}^{\infty}\int_{t_i}^{t_{i+1}}V_i(t)\sin\omega t\,dt$, and $I_3=\Sigma_{i=-\infty}^{\infty}\int_{t_i}^{t_{i+1}}V_i^2(t)\,dt$ contain the velocity of the unperturbed system V_i, corresponding to the separatrix solution $H_0=0$ with $f(t)=f_i$. The direct integration leads to the expression

$$V_i(t)=-\frac{\sqrt{2}(1-3\xi_i^2)^{3/2}}{\sqrt{1-\xi_i^2}} \times\frac{\sinh\sqrt{1-3\xi_i^2}t}{(\cosh\sqrt{1-3\xi_i^2}t+\sqrt{2}\xi_i/\sqrt{1-\xi_i^2})^2}, \tag{3}$$

where $\xi_i=-2\sqrt{3}\cos(\theta_i/3+\pi/3)$ and $\theta_i=\arccos 3\sqrt{3}f_i/2$. If D changes sign for any τ, separatrices intersect each other giving rise to homoclinic instability. We can introduce the order parameter

$$O=\frac{\alpha}{\gamma}\frac{I_3}{\sqrt{I_1^2+I_2^2}}, \tag{4}$$

which is a stochastic quantity. Equation (2) implies that an irregular behavior emerges if $O<1$. Thus the probability that the motion is regular, $P_{\rm reg}(\alpha,\gamma,\omega,k)$, is proportional to the number of cases satisfying the condition $O>1$.

We have calculated the parameter O, simulating quantities t_i and f_i for both AC ($R_D=0.06, L_D=0.2$) and EC ($R_D=0.25, L_D=0.2$) cases. An event in the statistical ensemble is determined by initial condition of the particle in the Sinai billiard. We have chosen initial conditions sampling the particle position uniformly inside the billiard and its momentum on the unit circle. Figure 1(a) shows the dependence of $P_{\rm reg}$ on the frequency ω. If the friction coefficient equals the amplitude of the driving force ($\alpha=\gamma$), the noiseless system exhibits a chaotic window around $\omega=1$. The EC noise slightly diminishes that window. On the other hand, for the AC noise the border between regularity and chaos becomes diffused and a stable behavior also takes place inside the noiseless chaotic window, though with small probability. The qualitative difference between both kinds of noise can also be observed if we start from the critical order parameter value ($O=1$) of the noiseless system, and gradually increase the noise amplitude k [Fig. 1(b)]. In contrast to the rapid turn toward the regular regime in the EC case, $P_{\rm reg}(k)$ rises slowly and saturates at a finite value for the AC noise. Therefore, the response of the system to small perturbation is more moderate for the algebraic noise correlations.

The condition of transverse intersection, $O<1$, indicates the appearance of local instability in the neighborhood of separatrices after applying a small perturbation. In general, it does not imply the existence of strange attractor and irregular motion in a substantial domain of the phase space. To study the fully chaotic case, we have calculated the largest Lyapunov exponents λ [13], determined from linearized equations of motion integrated along a noisy trajectory. Such procedure estimates the divergence rate of close trajectories subjected to the same realizations of the stochastic force. The Lyapunov exponent is a good measure of dynamical instability of systems with white noise, unless a strong intermittent behavior is present [14]. The AC noise, which does not act as isolated impulses but its nonzero values persist for a finite period of time, can be caused in practice by some collective mechanism. Then the assumption that close trajectories are affected by the same value of random force would be especially reasonable. For problems with noise, the Lyapunov exponent is a stochastic quantity, given by a probability distribution. Figure 2(a) presents such distributions for both kinds of noise correlation. The average of λ for the EC noise retains the same value as for the noise-free case ($\lambda=0.165$), and the distribution is narrow. Conversely, the AC noise reduces the Lyapunov exponent considerably, and this effect becomes stronger for larger noise amplitude k [Fig. 2(b)]. Finally, for sufficiently strong noise, $\langle\lambda\rangle$ falls to zero because the dynamics is dominated by long intervals of constant acceleration. On the other hand, the average Lyapunov exponent for the EC noise remains constant, and is equal to the noise-free value. The latter result agrees with the well known fact that in the chaotic regime the uncorrelated noise does not modify the Lyapunov exponent substantially [15].

One can ask how the character of motion, either regular or chaotic, is reflected on physical, experimentally measurable quantities. In Ref. [4] we studied the motion of a particle subjected to a quadratic (integrable) potential and both AC and EC noises, in particular the escape from a potential well and a passage over a barrier. Models of that kind are sup-

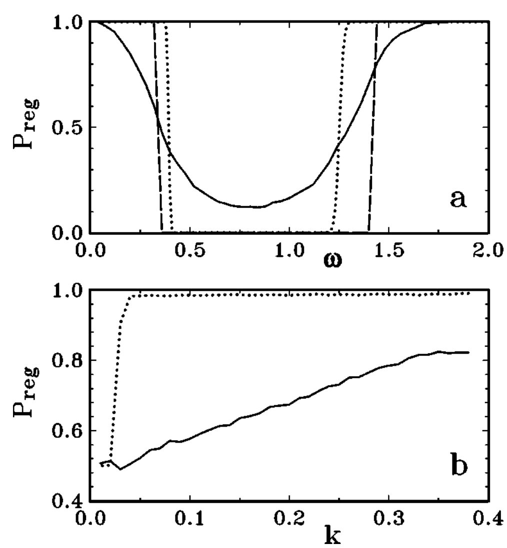

FIG. 1. Probability $P_{\rm reg}$ that the motion is homoclinically stable. The solid and dotted lines indicate the cases of AC and EC noise, respectively. (a) $P_{\rm reg}$ calculated as a function of ω with parameters $\alpha=\gamma$ and $k=0.3$. The noise-free result ($k=0$) is marked by the dashed line. (b) $P_{\rm reg}$ as a function of the noise amplitude k, with parameters $\alpha=3\sqrt{2}\pi\gamma\omega/4\cosh(\pi\omega/2)$, $\gamma=0.02$, and $\omega=1.5$, corresponding to the critical order parameter value ($O=1$) of the noiseless system.

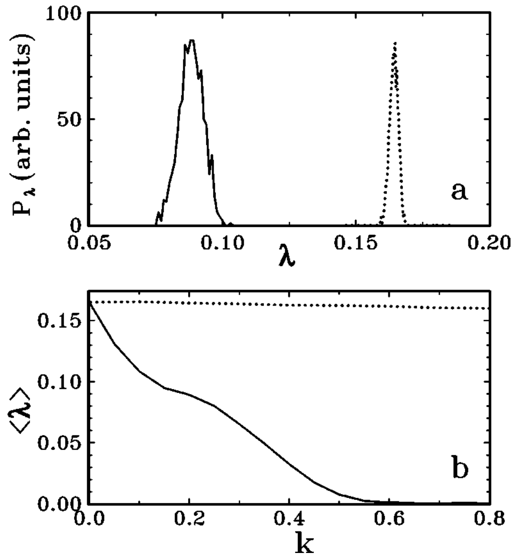

FIG. 2. (a) Distributions of the Lyapunov exponent calculated with parameters $\alpha=1$, $\gamma=2$, $\omega=1.5$, and $k=0.2$. (b) The average Lyapunov exponent as a function of k; other parameters are the same. The solid and dotted lines correspond to the cases of AC and EC noise, respectively.

posed to describe, e.g., the evaporation and fission processes in nuclear physics. We have found that the survival probability, defined by the number of stochastic trajectories that do not yet escape at a given time, has a tail proportional to $1/t$. The energy of particles, both escaping from the well and passing over the barrier, possesses a distribution with the Gaussian tail — a sign that equilibration has not been reached, even for long survival times. On the other hand, the shape of the energy distribution for the EC noise is exponential (Maxwellian). In the latter case, the equilibration results from frequent changes of the applied stochastic force, whereas the AC noise is connected with long intervals of constant acceleration. Within such intervals the motion becomes very regular, due to the balance between the noise and the conservative force [4]. Another characteristic feature of the energy distribution for the AC noise is the presence of a sharp peak, positioned at a relatively low energy, observed if the depth of potential well is not too large. The appearance of the peak is independent of a particular form of noise generator [16], and also takes place for the generalized Langevin equation, including the retarded friction term [17]. The peak corresponds to escaping particles subjected to only a single value of the stochastic force.

A similar study can be carried out with system (1). A double-well system with periodic perturbation and noise has been frequently applied to model a variety of problems, e.g., resonances in lasers [18], recurrences of the Earth's ice ages [19], and sensory neuron activity [20]. The consequence of chaos for the particle escape through a barrier, both with and without white noise, has been also studied [21]. In this paper, we will present a simple example of such process, with the AC noise, and compare results for regular and chaotic cases. For that purpose, we put the particle at the bottom of the right potential minimum, and compute the time the oscillatory force and the AC noise need to ''throw'' the particle above the barrier. Then the calculation is terminated. We also determine the total particle energy just after the barrier crossing, a quantity accessible experimentally for many physical problems. Two cases are considered: the chaotic case ($\alpha=0.001$, $\gamma=0.5$, $\omega=1$ and $k=1$), characterized by positive Lyapunov exponents for systems both with and without noise ($k=0$); and the regular case ($\alpha=0.3$, $\gamma=0.2$, $\omega=1$ and $k=0.2$), for which the respective exponents vanish. The survival probability has an algebraic $1/t$ tail in both cases, similarly as for the quadratic potential [4]. The energy distributions of particles at the barrier are juxtaposed in Fig. 3. The shape of the distribution is relatively smooth in the regular case [Fig. 3(b)], and the tail has a Gaussian shape. This result agrees with spectra obtained for the quadratic potential. The chaotic case, shown in Fig. 3(a), is different. Since the deterministic force can now randomize the motion, equilibration is easier and the tail turns to the exponential (Maxwellian) shape. Only for large energies does it bend down more sharply (not shown in the figure). The highest peak results from the existence of long free paths of particle in the billiard [4]—it corresponds to trajectories of the Brownian particle driven by a single constant value of the stochastic force. A peculiar consequence of chaoticity shows up at the low-energy region of the spectrum, where a complicated structure of many maxima develops. A full assessment of the experimental implications of differences between both kinds of spectra for specific physical processes needs individual studies.

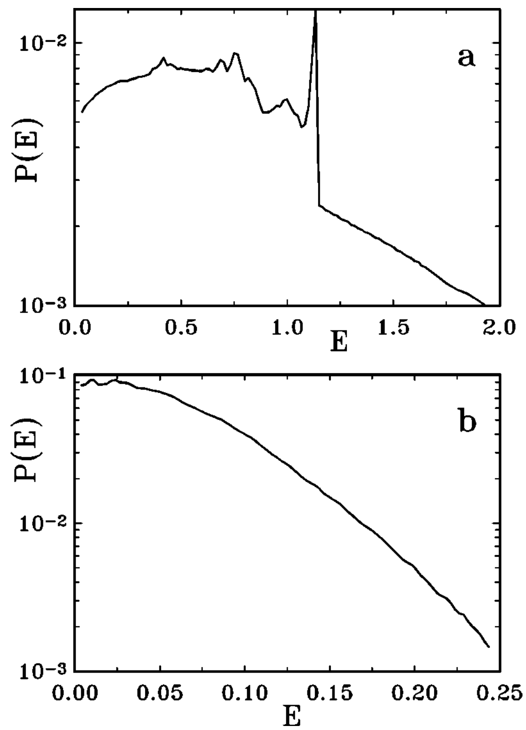

FIG. 3. Energy distribution of particles passing over the potential barrier for the AC noise. (a) The chaotic case ($\alpha=0.001$, $\gamma=0.5$, $\omega=1$, and $k=1$); the statistical error is comparable to the linewidth (8×10^6 trajectories calculated). The tail of the distribution (not shown in the figure) has an exponential shape. (b) The regular case ($\alpha=0.3$, $\gamma=0.2$, $\omega=1$, and $k=0.2$); the statistical ensemble comprises 10^6 trajectories. Both distributions are normalized to unity.

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